Slopes of modular forms and ghost conjecture

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Introduction

Slopes of cuspforms

- Fix a prime $p > 2$ and a positive integer N.
- Let $f \in \mathrm{S}_k(\Gamma_1(N))$ be a cuspidal modular form of weight $k \geq 2$ and level $\Gamma_1(N)$. Let $f = \sum\limits_{n \geq 1} a_n q^n$, $q = e^{2\pi i z}$ be the Fourier expansion of $f(z)$. Assume that f is normalized, i.e. $a_1 = 1$.
- Fix two embeddings $i : \bar{\mathbb{Q}} \to \mathbb{C}$ and $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$. Let $v_p(\cdot)$ be the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. Let $|\cdot|_p = p^{-v_p(\cdot)}$ be the corresponding p-adic norm.
- When f is an Hecke eigenform, Shimura proved that the subfield $\mathbb{Q}(f)$ of $\mathbb{C}$ generated by the coefficients a_n's is a finite extension of $\mathbb{Q}$. We will view a_n's as p-adic numbers in $\mathbb{C}_p$ via the embeddings i and i_p. The p-adic valuation $v_p(a_p)$ is called the (p-adic) slope of f.
- Goal: give an algorithm to compute the slopes of Hecke eigenforms.

Introduction

Newton polygon of U_p-operator

- Assume $p|N$. Let $\det(\mathrm{I} - X \cdot \mathrm{U}_p|_{\mathrm{S}_k(\Gamma_1(N))}) = \sum_{i=0}^{d} c_i X^i$ be the characteristic polynomial of the U_p-operator on the space $\mathrm{S}_k(\Gamma_1(N))$ with $d = \dim \mathrm{S}_k(\Gamma_1(N))$. The Newton polygon of this polynomial is the lower convex hull of the points $(i, v_p(c_i))$, $i = 0, \ldots, d$ on the x-y plane.
- Since a_p is the U_p-eigenvalue of f, the computation of the slopes of Hecke eigenforms is equivalent to that of the slopes of the Newton polygon of $\det(\mathrm{I} - X \cdot \mathrm{U}_p)$.
- Let $\mathrm{S}_k^\dagger(\Gamma_1(N))$ be the space of overconvergent cuspidal modular forms of weight k and level $\Gamma_1(N)$. The U_p-operator on $\mathrm{S}_k(\Gamma_1(N))$ extends to a compact operator on $\mathrm{S}_k^\dagger(\Gamma_1(N))$ and let $C_k(X) = \det(\mathrm{I} - X \cdot \mathrm{U}_p|_{\mathrm{S}_k^\dagger(\Gamma_1(N))}) = \sum_{n \geq 0} c_n X^n \in \mathbb{Q}_p[\![X]\!]$ be its Fredholm series. We can define the Newton polygon $\mathrm{NP}(C_k)$ of $C_k(X)$ similarly.

- We define the weight space $\mathcal{W}$ to be the rigid analytic space associated to the Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$. Geometrically $\mathcal{W}$ is the disjoint union $\bigsqcup_{\varepsilon} \mathcal{W}_\varepsilon$ of open unit discs indexed by the characters of the torsion subgroup Δ of $\mathbb{Z}_p^\times$. A p-adic weight κ is a closed point $\kappa \in \mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. Fix a topological generator $\gamma = \exp(p)$ of $1 + p\mathbb{Z}_p$. We define $w_\kappa = \kappa(\gamma) - 1$ to be the coordinate of κ on the corresponding weight disc.
- For any p-adic weight κ, let $\mathrm{S}_\kappa^\dagger(\Gamma_1(N))$ be the space of overconvergent cuspidal modular forms of weight κ and level $\Gamma_1(N)$. We can define the Fredholm series $C_\kappa(X)$ for the U_p-operator in weight κ in the same way.
- Coleman proved that for each character ε of Δ, there exists a two variable series $C^{(\varepsilon)}(w, X) = 1 + \sum_{n \geq 1} c_n^{(\varepsilon)}(w) X^n \in \mathbb{Z}_p[\![w, X]\!]$, such that $C^{(\varepsilon)}(w_\kappa, X) = C_\kappa(X)$ for all $\kappa \in \mathcal{W}_\varepsilon(\mathbb{C}_p)$.
- Bergdall and Pollack's idea: find an explicit 'model' to approximate $C^{(\varepsilon)}(w, X)$.

Ghost conjecture

Galois input

- From now on we assume $(p, N) = 1$. Denote $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$.
- Let $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a continuous irreducible odd representation.
- Assume that $\bar{\rho}$ is modular of level N. For every integer $k \geq 2$, let $\mathrm{S}_k(\Gamma_1(N))_{\bar{\rho}} := \big(\mathrm{S}_k(\Gamma_1(N), \mathbb{Z}_p)_{\mathfrak{m}_{\bar{\rho}}}\big)[\frac{1}{p}]$ be the $\bar{\rho}$-component of $\mathrm{S}_k(\Gamma_1(N), \mathbb{Q}_p)$. We define $\mathrm{S}_k(\Gamma)_{\bar{\rho}}$ in a similar way. The space $\mathrm{S}_k(\Gamma_1(N))_{\bar{\rho}}$ is stable under the T_p-operator and $\mathrm{S}_k(\Gamma)_{\bar{\rho}}$ is stable under the U_p-operator.
- Fix a decomposition group $D_p \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and let $I_p \subset D_p$ be the inertia subgroup. Let $\omega_1 : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \bar{\mathbb{F}}_p$ be the mod p cyclotomic character.
- Assumption (Buzzard regularity): the local representation $\bar{\rho}|_{D_p}$ is reducible ,and there exists $a \in \{1, \dots, p-4\}$ and $b \in \{0, \dots, p-2\}$, such that

$$\bar{\rho}|_{I_p} \sim \begin{pmatrix} \omega_1^{a+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega_1^b$$

- Under this regular assumption, the T_p-eigenvalues on the space $\mathrm{S}_k(\Gamma_1(N))_{\bar{\rho}}$ are all p-adic units for $2 \leq k \leq 1+p$.

Ghost conjecture

ghost series

- For any p-adic weight $\kappa \in \mathcal{W}(\mathbb{C}_p)$, let $\mathrm{S}^\dagger_\kappa(\Gamma)$ denote the space of overconvergent cuspforms of weight κ and tame level N and $\mathrm{S}^\dagger_\kappa(\Gamma)_{\bar\rho}$ be its $\bar\rho$-isotypic subspace. The U_p-operator leaves $\mathrm{S}^\dagger_\kappa(\Gamma)_{\bar\rho}$ and we let $C_{\bar\rho,\kappa}(X) = C_{\bar\rho,\kappa,N}(X) = 1 + \sum_{i\geq 1} c_i(w_\kappa)X^i$ be the Fredholm series of this operator.
- Let $k_0 = 2 + a + 2b$. For each integer $k \geq 2$ satisfying $k \equiv k_0 \bmod (p-1)$, we define $d_k^{\mathrm{ur}} = \dim \mathrm{S}_k(\Gamma_1(N))_{\bar\rho}, d_k^{\mathrm{Iw}} = \dim \mathrm{S}_k(\Gamma)_{\bar\rho}$ and $d_k^{\mathrm{new}} = d_k^{\mathrm{Iw}} - 2d_k^{\mathrm{ur}}$.
- For every k as above, we define a sequence $\{m_i(k) | i \geq 1\}$ of integers as follows:

$$m_i(k) = \begin{cases} \min\{i - d_k^{\mathrm{ur}}, d_k^{\mathrm{Iw}} - d_k^{\mathrm{ur}} - i\}, & \text{for } d_k^{\mathrm{ur}} < i < d_k^{\mathrm{Iw}} - d_k^{\mathrm{ur}}, \\ 0, & \text{otherwise }. \end{cases}$$

  Explicitly, the sequence $\{m_i(k) | i \geq 1\}$ is given by the following palindromic pattern

$$\underbrace{0, \ldots, 0}_{d_k^{\mathrm{ur}}}, 1, 2, 3, \ldots, \tfrac{1}{2}d_k^{\mathrm{new}} - 1, \tfrac{1}{2}d_k^{\mathrm{new}}, \tfrac{1}{2}d_k^{\mathrm{new}} - 1, \ldots, 3, 2, 1, 0, 0, \ldots,$$

Ghost conjecture

ghost series

- For every integer k, we define an algebraic weight $\kappa_k : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, $z \mapsto z^{k-2}$ with coordinate $w_k = w_{\kappa_k} = \exp(p(k-2)) - 1$. We let $\mathcal{W}_k$ denote the weight disc which κ_k belongs to.
- For $i \geq 1$, we define

$$g_i(w) = \prod_{k \geq 2, k \equiv k_0 \bmod (p-1)} (w - w_k)^{m_i(k)} \in \mathbb{Z}_p[w].$$

Definition

We define the ghost series for $\bar{\rho}$ to be the formal power series

$$G_{\bar{\rho}}(w, X) = 1 + \sum_{i=1}^{\infty} g_i(w) X^i \in \mathbb{Z}_p[w][\![X]\!].$$

Ghost conjecture

statement

Conjecture (Bergdall-Pollack's ghost conjecture)

For any p-adic weight $\kappa \in \mathcal{W}_{k_0}(\mathbb{C}_p)$, we have $\mathrm{NP}(C_{\bar{\rho},\kappa}(X)) = \mathrm{NP}(G_{\bar{\rho}}(w_\kappa, X))$.

- The Buzzard regular assumption on $\bar{\rho}$ is essential. The ghost conjecture is false without this assumption.

Theorem (Liu-Truong-Xiao-Z.)

Assume $p \geq 11$ and $2 \leq a \leq p-5$. Then the ghost conjecture is true.

Intuition on ghost zeroes

analysis on slopes of classical modular forms

- Let $f \in \mathrm{S}_k(\Gamma)$ be a normalized Hecke eigenform. Let $\varepsilon(p)$ be the eigenvalue of f for the diamond operator $\langle p \rangle$. We have the following facts about the slopes of classical modular forms:
  1. When f is new at p, we have $a_p^2 = \varepsilon(p)p^{k-2}$ and hence f has slope $\frac{k-2}{2}$;
  2. The other U_p-eigenvalues in $\mathrm{S}_k(\Gamma)$ come in pairs: for a normalized eigenform $g \in \mathrm{S}_k(\Gamma_1(N), \varepsilon)$ with T_p-eigenvalue a_p, it has two p-stabilizations $f_\alpha(z) = f(z) - \beta f(pz)$, $f_\beta(z) = f(z) - \alpha f(pz)$ in $\mathrm{S}_k(\Gamma)$ with U_p-eigenvalues α and β, where α, β are the roots of $X^2 - a_p X + \varepsilon(p)p^{k-1}$. So the slopes of these two p-old forms sum to $k-1$. In particular, if $v_p(a_p)$ can be read off from $v_p(\alpha)$ and $v_p(\beta)$;
- The slopes of p-oldforms behave very different from those of p-newforms. Let $f \in \mathrm{S}_k(\Gamma_1(N))_{\bar{\rho}}$ be a eigenform with T_p-eigenvalue a_p. Berger-Li-Zhu proved that $v_p(a_p) \leq \lfloor \frac{k-2}{p-1} \rfloor$ (conjecturally this can be strengthened to $\lfloor \frac{k-2}{p+1} \rfloor$);

Intuition on ghost zeroes

Newton polygon of U_p-operator on the space of classical modular forms

Conjecture (Gouvěa)

For each k, write $\alpha_1(k), \ldots, \alpha_d(k)$ for the list of U_p-slopes on $\mathrm{S}_k(\Gamma_0(Np))$, and let μ_k denote the uniform probability measure of the multiset $\{\frac{\alpha_1(k)}{k-1}, \ldots, \frac{\alpha_d(k)}{k-1}\} \subset [0,1]$. Then the measure μ_k's weakly converge to $\frac{1}{p+1}\delta_{[0,\frac{1}{p+1}]} + \frac{1}{p+1}\delta_{[\frac{p}{p+1},1]} + \frac{p-1}{p+1}\delta_{\frac{1}{2}}$, where $\delta_{[a,b]}$ denotes the uniform probability measure on the interval $[a,b]$, and $\delta_{\frac{1}{2}}$ is the Dirac measure at $\frac{1}{2}$.

- For any $k \geq 2$ with $k \equiv k_0 \bmod (p-1)$, the Newton polygon of U_p-operator on $\mathrm{S}_k(\Gamma)_{\bar{\rho}}$ should have a line segment of length d_k^{new} and slope $\frac{k-2}{2}$. In particular, the point $(i, v_p(c_i(w_k)))$ is not a vertex of $\mathrm{NP}(C_{\bar{\rho},\kappa_k}(X))$, for $i = d_k^{\mathrm{ur}}+1, \ldots, d_k^{\mathrm{ur}}+d_k^{\mathrm{new}}-1$.
- The above integers i's are exactly those integers with the property $g_i(w_k) = 0$.

Intuition on ghost multiplicities

automorphic forms on definite quaternion algebras I

- Let $\mathbb{A}_f$ be the ring of finite adeles and $\mathbb{A}_f^{(p)}$ be the subring of finite prime-to-p adeles.
- Let D be a definite quaternion algebra over $\mathbb{Q}$ and we assume that D is split at p. Set $D_f = D \otimes_{\mathbb{Q}} \mathbb{A}_f$. Fix an open compact subgroup K^p of $(D \otimes \mathbb{A}_f^{(p)})^\times$. Let $\mathrm{Iw}_p = \left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right)$ be the Iwahori subgroup of $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.
- For every integer $k \geq 2$, let $\mathbb{Q}_p[z]^{\deg \leq k-2}$ be the space of polynomials of degree $\leq k-2$ over $\mathbb{Z}_p$. It carries a right action of the monoid $\mathbf{M}_1 = \left\{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}_p)^{\det \neq 0} | p | \gamma, p \nmid \delta\right\}$ given by

$$h|_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}(z) = (\gamma z + \delta)^{k-2} h\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbf{M}_1 \text{ and } h(z) \in \mathbb{Q}_p[z]^{\deg \leq k-2}.$$

- For $k \geq 2$, define the space of classical automorphic forms on D as:
$\mathrm{S}_k^D(K^p K_p) = \mathrm{Hom}_{K_p}(D^\times \backslash D_f^\times / K^p, \mathbb{Q}_p[z]^{\deg \leq k-2})$
$\mathrm{S}_k^D(K^p \mathrm{Iw}_p) = \mathrm{Hom}_{\mathrm{Iw}_p}(D^\times \backslash D_f^\times / K^p, \mathbb{Q}_p[z]^{\deg \leq k-2})$

Intuition on ghost multiplicities

automorphic forms on definite quaternion algebras II

- Fix $k \geq 2$, set $d_k^{\mathrm{ur}} = \dim \mathrm{S}_k^D(K^p K_p)$ and $d_k^{\mathrm{Iw}} = \mathrm{S}_k^D(K^p \mathrm{Iw}_p)$. Define $d_k^{\mathrm{new}} = d_k^{\mathrm{Iw}} - 2d_k^{\mathrm{ur}}$ as before.
- Fix a decomposition of the double coset

$$\mathrm{Iw}_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{Iw}_p = \bigsqcup_{j=0}^{p-1} \mathrm{Iw}_p v_j, \text{ for } v_j = \left(\begin{smallmatrix} p & 0 \\ pj & 1 \end{smallmatrix}\right), j = 0, \ldots, p-1.$$

We define the U_p-operator on $\mathrm{S}_k^D(K^p \mathrm{Iw}_p)$ via the formula

$$\mathrm{U}_p(\varphi)(x) = \sum_{j=0}^{p-1} \varphi(x v_j^{-1})|_{v_j} \text{ for } \varphi \in \mathrm{S}_k^D(K^p \mathrm{Iw}_p), x \in D_f^\times.$$

Intuition on ghost multiplicities

families of p-adic automorphic forms

- Fix a character ε. Let $\kappa : 1 + p\mathbb{Z}_p \to \mathbb{Z}_p[\![w]\!]^\times$, $\exp(p) \mapsto 1 + w$ be the universal character and $\mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p[\![w]\!])^{(\varepsilon)}$ be the space of continuous maps from $\mathbb{Z}_p$ to $\mathbb{Z}_p[\![w]\!]$, endowed with a right action of the monoid $\mathbf{M}_1$ given by

$$h|_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}(z) = \varepsilon(\bar{\delta})\kappa\left(\frac{\gamma z + \delta}{\omega(\bar{\delta})}\right)h\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \ h \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p[\![w]\!])^{(\varepsilon)}.$$

- We define the space of p-adic automorphic forms on $\mathcal{W}_\varepsilon$ to be

$$\mathrm{S}^{(\varepsilon)}_{p\text{-}adic}(K^p \mathrm{Iw}_p) = \mathrm{Hom}_{\mathrm{Iw}_p}(D^\times \setminus D_f^\times / K^p, \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p[\![w]\!])^{(\varepsilon)}),$$

  and define the U_p operator on it by the same formula as above.

- When K^p is sufficiently small, the space $\mathrm{S}^{(\varepsilon)}_{p\text{-}adic}(K^p \mathrm{Iw}_p)$ is a finite direct sum of $\mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p[\![w]\!])$.

Intuition on ghost multiplicities

p-stabilization process

- We define a map

$$\mathrm{AL}_k : \mathrm{S}_k^D(K^p\mathrm{Iw}_p) \to \mathrm{S}_k^D(K^p\mathrm{Iw}_p), \varphi \mapsto \Big(\mathrm{AL}_k(\varphi)(x) = \varphi\big(x\big(\begin{smallmatrix} 0 & p^{-1} \\ 1 & 0 \end{smallmatrix}\big)\big)|_{\big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)}\Big), x \in D_f^\times.$$

  AL_k is called the Atkin-Lehner involution on $\mathrm{S}_k^D(K^p\mathrm{Iw}_p)$.

- Define two embeddings $\iota_1, \iota_2 : \mathrm{S}_k^D(K^pK_p) \to \mathrm{S}_k^D(K^p\mathrm{Iw}_p)$ as

$$\iota_1(\psi) = \psi, \iota_2(\psi) = \mathrm{AL}_k \circ \iota_1(\psi) \text{ for } \psi \in \mathrm{S}_k^D(K^pK_p).$$

- Fix a set $\{u_j = \big(\begin{smallmatrix} 1 & 0 \\ j & 1 \end{smallmatrix}\big), j = 0, \ldots, p-1, u_\star = \big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big)\}$ of coset representatives of $\mathrm{Iw}_p \setminus K_p$. Define a map $\mathrm{proj}_1 : \mathrm{S}_k^D(K^p\mathrm{Iw}_p) \to \mathrm{S}_k^D(K^pK_p)$ as

$$\mathrm{proj}_1(\varphi)(x) = \sum_{j=0,\ldots,p-1,\star} \varphi(xu_j)|_{u_j^{-1}}$$

Intuition on ghost multiplicities

a corank result

- Key observation: for any $\varphi \in \mathrm{S}_k^D(K^p\mathrm{Iw}_p)$, we have $\mathrm{U}_p(\varphi) = \iota_2(\mathrm{proj}_1(\varphi)) - \mathrm{AL}_k(\varphi)$. In other words, we have an equality

$$\mathrm{U}_p = \iota_2 \circ \mathrm{proj}_1 - \mathrm{AL}_k.$$

- We can find a basis of the space $\mathrm{S}_k^D(K^p\mathrm{Iw}_p)$, such that the matrix $M(\mathrm{AL}_k)$ of the Atkin-Lehner involution AL_k under this basis is antidiagonal. Let M and M_1 be the matrix of the operators U_p and $\iota_2 \circ \mathrm{proj}_1$ uner the above basis. Then the rank of M_1 is d_k^{ur} and we have $M = M_1 - M(\mathrm{AL}_k)$.
- For any $i \times i$ matrix N, we define its corank to be the integer $i - \mathrm{rank}(N)$. For $1 \le i \le d_k^{\mathrm{Iw}}$, let $M(i)$ be the upper left $i \times i$-submatrix of M. We have

$$\text{corank of } \; M(i) \ge \begin{cases} i - d_k^{\mathrm{ur}} & \text{for } i \le \frac{1}{2}d_k^{\mathrm{Iw}}, \\ i - (d_k^{\mathrm{ur}} + 2(i - \frac{1}{2}d_k^{\mathrm{Iw}})), & \text{for } i > \frac{1}{2}d_k^{\mathrm{Iw}}. \end{cases}$$

Intuition on ghost multiplicities

halo bound

- In summary, we have

$$\text{corank of } \ M(i) \geq m_i(k), \text{ for all } 1 \leq i \leq d_k^{\mathrm{Iw}}.$$

- There exists another basis of the space $\mathrm{S}^{(\varepsilon)}_{p\text{-}adic}(K^p\mathrm{Iw}_p)$ such that the matrix $M' = (m'_{i,j})$ of the U_p-operator under this basis satisfies the halo bound $v_p(m'_{i,j}) \geq j - \lfloor \frac{j}{p} \rfloor$.
- By Coleman's result, for every component $\mathcal{W}_\varepsilon$ of the weight space, we have a two variable power series $C^{(\varepsilon)}(w, X) = 1 + \sum\limits_{n \geq 1} c_n(w) X^n \in \mathbb{Z}_p[\![w, X]\!] \subset \mathcal{O}(\mathcal{W}_\varepsilon)[\![X]\!]$, such that $C^{(\varepsilon)}(w_k, X)$ is the Fredholm series of the U_p-operator on $\mathrm{S}_k^{D,\dagger}(K^p\mathrm{Iw}_p)$ for $\kappa_k \in \mathcal{W}_\varepsilon$.
- We can write $c_n(w) = g_n(w) \cdot h(w) + \mathrm{Err}(w)$, where the 'error' term $\mathrm{Err}(w)$ has large p-adic valuation.

Applications

automorphic applications I

Conjecture (Gouvěa)

For each k, write $\alpha_1(k), \ldots, \alpha_d(k)$ for the list of U_p-slopes on $\mathrm{S}_k(\Gamma_0(Np))$, and let μ_k denote the uniform probability measure of the multiset $\{\frac{\alpha_1(k)}{k-1}, \ldots, \frac{\alpha_d(k)}{k-1}\} \subset [0,1]$. Then the measure μ_k's weakly converge to $\frac{1}{p+1}\delta_{[0,\frac{1}{p+1}]} + \frac{1}{p+1}\delta_{[\frac{p}{p+1},1]} + \frac{p-1}{p+1}\delta_{\frac{1}{2}}$, where $\delta_{[a,b]}$ denotes the uniform probability measure on the interval $[a,b]$, and $\delta_{\frac{1}{2}}$ is the Dirac measure at $\frac{1}{2}$.

- Bergdall and Pollack proved that their ghost conjecture implies Gouvěa's conjecture for the $\bar{\rho}$-component of $\mathrm{S}_k(\Gamma_0(Np))$.

Conjecture (Gouvêa–Mazur)

There is a function $M(n)$ linear in n such that if $k_1, k_2 > 2n+2$ and $k_1 \equiv k_2 \bmod (p-1)p^{M(n)}$, then the sequences of U_p-slopes (with multiplicities) on $\mathrm{S}_{k_1}(\Gamma_0(Np))$ and $\mathrm{S}_{k_2}(\Gamma_0(Np))$ agree up to slope n.

- Rufei Ren proved that the ghost conjecture implies that Gouvêa–Mazur conjecture holds if we only consider the $\bar{\rho}$-component of the spaces of modular forms.

Applications

Application to eigencurves

- For $r \in (0,1)$, we use $\mathcal{W}^{>r}$ to denote the union of annuli where the parameter $|w|_p > r$.
- Let $\mathcal{C}$ be the eigencurve of tame level N. Every close point of $\mathcal{C}$ corresponds to a finite slope normalized overconvergent eigenform. Let $\mathrm{wt}: \mathcal{C} \to \mathcal{W}$ be the weight map. We use $\mathcal{C}^{>r}$ to denote the preimage $\mathrm{wt}^{-1}(\mathcal{W}^{>r})$.

Conjecture

When r is sufficiently close to 1, the following statements hold:

1. *The space $\mathcal{C}^{>r}$ is a disjoint union of connected components $Z_1, Z_2, \ldots$ such that $\mathrm{wt}: Z_n \to \mathcal{W}^{>r}$ is finite and flat;*
2. *There exist a sequence of rational numbers $\alpha_1 \leq \alpha_2 \leq \ldots$ such that for all n and $z \in Z_n$, we have $v_p(a_p(z)) = \alpha_n v_p(w_{\mathrm{wt}(z)})$;*
3. *The sequence $\alpha_1, \alpha_2, \ldots$ is a disjoint union of finitely many arithmetic progressions, counted with multiplicity.*

- Let $f = \sum_{n \geq 1} a_n q^n \in \mathrm{S}_k(\Gamma_0(N))$ be a normalized Hecke eigenform. Let $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be the p-adic Galois representation associated to f.
- Let $\rho_{f,p}$ be the restriction of ρ_f to the p-decomposition group. Assume $v_p(a_p) > 0$ and $a_p^2 \neq 4p^{k-1}$. The local representation $\rho_{f,p}$ is crystalline and is determined by the pair (k, a_p).

Conjecture (Breuil-Buzzard-Gee)

If the mod p reduction $\bar{\rho}_{f,p}$ of $\rho_{f,p}$ is reducible, the slope $v_p(a_p)$ belongs to $\mathbb{Z}$.

- We proved that ghost conjecture implies Breuil-Buzzard-Gee conjecture.

A final remark

local ghost conjecture

- Let $\bar{\rho}_p : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a reducible local Galois representation. Assume that $p \geq 11$ and there exists $a \in \{2, \ldots, p-5\}$ and $b \in \{0, \ldots, p-2\}$, such that

$$\bar{\rho}_p|_{I_p} \sim \begin{pmatrix} \omega_1^{a+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega_1^b$$

- Let $k \geq 2$ be an integer. For a crystalline lift V_{k,a_p} of $\bar{\rho}_p$ of Hodge-Tate weights $(0, k-1)$, let a_p be the trace of crystalline Frobenius on the (weakly) admissible module corresponding to V_{k,a_p}. In the joint work with Ruochuan Liu, Nha Truong and Liang Xiao, we prove a local version of ghost conjecture which gives an algorithm to compute $v_p(a_p)$. In particular, we prove that $v_p(a_p) \in \mathbb{Z}$ when k is even, and $v_p(a_p) \in \frac{1}{2}\mathbb{Z}$ when k is odd.

Thank you!