# A Quillen-Lichtenbaum Conjecture for Dirichlet $L$-functions

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December 17, 2022

Joint work in progress with Elden Elmanto

1. Review of classical Quillen-Lichtenbaum Conjecture for Dedekind zeta functions

2. Rational equivariant algebraic $K$-theory of number fields

3. Quillen-Lichtenbaum for Dirichlet $L$-functions

# Review of classical Quillen-Lichtenbaum Conjecture for Dedekind zeta functions

# The Riemann and Dedekind zeta functions

## Definition

Let $\mathbb{F}$ be a number field. The **Dedekind zeta function** attached to $\mathbb{F}$ is:

$$\zeta_{\mathbb{F}}(s) = \sum_{(0)\neq\mathcal{I}\trianglelefteq\mathcal{O}_{\mathbb{F}}} \frac{1}{|\mathcal{O}_{\mathbb{F}}/\mathcal{I}|^s} = \prod_{(0)\neq\mathfrak{p}\trianglelefteq\mathcal{O}_{\mathbb{F}}} \frac{1}{1-|\mathcal{O}_{\mathbb{F}}/\mathfrak{p}|^{-s}}.$$

The Dedekind zeta function attached to $\mathbb{Q}$ is the **Riemann zeta function**.

- Converges when $\mathrm{Re}(s) > 1$, admits analytic continuation to $\mathbb{C}\backslash\{1\}$.
- Special value: $\zeta_{\mathbb{F}}(1-n) \in \mathbb{Q}$.

$$\mathrm{ord}_{s=1-n}\zeta_{\mathbb{F}}(s) = \begin{cases} r_1 + r_2 - 1, & n = 1; \\ r_1 + r_2, & n > 1 \text{ odd}; \\ r_2, & n \text{ even}. \end{cases}$$

- Functional equation: $\widehat{\zeta_{\mathbb{F}}}(s) = \widehat{\zeta_{\mathbb{F}}}(1-s)$.
- Riemann Hypothesis: $\zeta_{\mathbb{F}}(s) = 0 \overset{??}{\Longrightarrow} s = 1-n$ or $\mathrm{Re}(s) = 1/2$.

# Two classical theorems in algebraic number theory

### Theorem (Dirichlet Unit Theorem)

*The group of units in $\mathcal{O}_{\mathbb{F}}$ is a finitely generated abelian group.*

$$\mathcal{O}_{\mathbb{F}}^{\times} \cong \mu(\mathbb{F}) \times \mathbb{Z}^{\oplus r_1+r_2-1} \Longrightarrow \dim_{\mathbb{Q}} \mathcal{O}_{\mathbb{F}}^{\times} \otimes \mathbb{Q} = \mathrm{ord}_{s=0}\zeta_{\mathbb{F}}(s),$$

*where $\mu(\mathbb{F})$ is the group of roots of unity in $\mathbb{F}$.*

### Theorem (Class Number Formula)

$$\zeta_{\mathbb{F}}^{*}(0) = -\frac{|\mathrm{cl}(\mathbb{F})|}{|\mu(\mathbb{F})|} \cdot R(\mathbb{F}),$$

*where*

- $\zeta_{\mathbb{F}}^{*}(0)$ *is the leading coefficient of the Taylor series of* $\zeta_{\mathbb{F}}$ *at* $s = 0$.
- $\mathrm{cl}(\mathbb{F}) := \mathrm{Pic}(\mathcal{O}_{\mathbb{F}})$ *is the ideal class group of* $\mathbb{F}$.
- $R(\mathbb{F})$ *is the Dirichlet regulator of* $\mathbb{F}$.

# Reformulation in algebraic $K$-theory

The first two algebraic $K$-groups of $R = \mathcal{O}_{\mathbb{F}}$ are:

- $K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R) \cong \mathbb{Z} \oplus \mathrm{cl}(\mathbb{F})$.
- $K_1(R) \cong \mathrm{GL}(R)/E(R)$. Bass-Milnor-Serre showed $K_1(\mathcal{O}_{\mathbb{F}}) \cong \mathcal{O}_{\mathbb{F}}^{\times}$.

## Reformulation of the two theorems

- Dirichlet Unit Theorem:

$$\dim_{\mathbb{Q}} K_1(\mathcal{O}_{\mathbb{F}}) \otimes \mathbb{Q} = \mathrm{ord}_{s=0}\zeta_{\mathbb{F}}(s).$$

- Class Number Formula:

$$\zeta_{\mathbb{F}}^*(0) = -\frac{|K_0(\mathcal{O}_{\mathbb{F}})_{\mathrm{tors}}|}{|K_1(\mathcal{O}_{\mathbb{F}})_{\mathrm{tors}}|} \cdot R(\mathbb{F}).$$

# Algebraic $K$-groups of number fields

## Theorem (Dirichlet, Quillen-Borel)

*The algebraic $K$-groups $K_n(\mathcal{O}_{\mathbb{F}})$ are all finitely generated. More precisely, $K_{2n}(\mathcal{O}_{\mathbb{F}})$ is a finite abelian group when $n \geq 1$. For $K_{2n-1}(\mathcal{O}_{\mathbb{F}})$, we have:*

$$\dim_{\mathbb{Q}} K_{2n-1}(\mathcal{O}_{\mathbb{F}}) \otimes \mathbb{Q} = \begin{cases} r_1 + r_2 - 1, & n = 1; \\ r_1 + r_2, & n > 1 \text{ odd}; \\ r_2, & n \text{ even}, \end{cases} = \mathrm{ord}_{s=1-n}\zeta_{\mathbb{F}}(s).$$

## Theorem (Quillen-Lichtenbaum Conjecture, Voevodsky-Rost)

*The following identity*

$$\zeta_{\mathbb{F}}^{*}(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_{\mathbb{F}})|}{|K_{2n-1}(\mathcal{O}_{\mathbb{F}})_{\mathrm{tors}}|} \cdot R_n^B(\mathbb{F})$$

*holds up to powers of $2$, where $R_n^B(\mathbb{F})$ is the Borel regulator of $\mathbb{F}$.*

# From zeta functions to algebraic $K$-theory

Fix a prime $p$.

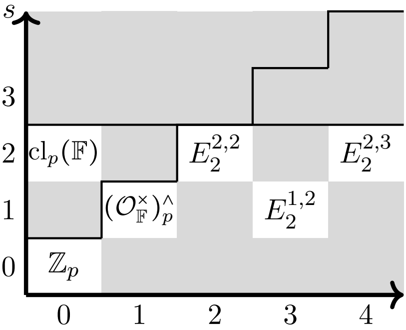
$$\begin{tikzcd}
\text{Frobenius action on } H^1_{\acute{e}t}(\mathcal{O}_{\mathbb{F}}[1/p,\zeta_{p^\infty}],\mathbb{Z}_p(t)) \arrow[d, squiggly] & & p\text{-adic } L\text{-function } \zeta^*_{\mathbb{F}}(1-t) \arrow[ll, squiggly, "\text{Iwasawa Main Conjecture}", "\text{Mazur-Wiles}"'] \\
H^r_c\left(\mathbb{Z}_p^\times; H^s_{\acute{e}t}(\mathcal{O}_{\mathbb{F}}[1/p,\zeta_{p^\infty}],\mathbb{Z}_p(t))\right) \arrow[d, Rightarrow, "\text{Hochschild-Lyndon-Serre SS}"] & & \\
H^{r+s}_{\acute{e}t}(\mathcal{O}_{\mathbb{F}}[1/p],\mathbb{Z}_p(t)) \arrow[rr, Rightarrow, "\text{Thomason SS}"] & & \pi_{2t-r-s}\left(L_{K(1)}K(\mathcal{O}_{\mathbb{F}}[1/p])\right) \\
H^{r+s}_{\mathrm{mot}}(\mathcal{O}_{\mathbb{F}}[1/p],\mathbb{Z}_p(t)) \arrow[u, "\text{étale-motivic comparison}"'] \arrow[rr, Rightarrow, "\text{Motivic SS}"] & & \pi_{2t-r-s}K(\mathcal{O}_{\mathbb{F}}[1/p])^\wedge_p \arrow[u, "L_{K(1)}"']
\end{tikzcd}$$

- Lichtenbaum: IMC implies an étale cohomology version of QLC.
- All spectral sequences above collapse at $E_2$-pages when $p > 2$.
- Voevodsky-Rost: the comparison map is an isomorphism when $t \geq r + s$.

Assume $\mathbb{F}$ is totally real for simplicity. When $n \geq 1$ and $p > 2$,

$$\zeta_{\mathbb{F}}(1-2n) \sim_p \pm \frac{|H^2_{\acute{e}t}(\mathcal{O}_{\mathbb{F}}[1/p], \mathbb{Z}_p(2n))|}{|H^1_{\acute{e}t}(\mathcal{O}_{\mathbb{F}}[1/p], \mathbb{Z}_p(2n))_{\mathrm{tors}}|} = \pm \frac{|H^2_{\mathrm{mot}}(\mathcal{O}_{\mathbb{F}}[1/p], \mathbb{Z}_p(2n))|}{|H^1_{\mathrm{mot}}(\mathcal{O}_{\mathbb{F}}[1/p], \mathbb{Z}_p(2n))_{\mathrm{tors}}|}$$
$$= \pm \frac{|K_{4n-2}(\mathcal{O}_{\mathbb{F}}[1/p]; \mathbb{Z}_p)|}{|K_{4n-1}(\mathcal{O}_{\mathbb{F}}[1/p]; \mathbb{Z}_p)_{\mathrm{tors}}|} \sim_p \pm \frac{|K_{4n-2}(\mathcal{O}_{\mathbb{F}})|}{|K_{4n-1}(\mathcal{O}_{\mathbb{F}})_{\mathrm{tors}}|}.$$

The motivic spectral sequence in Adams grading: (shaded $= 0$)




$E_2^{s,t} = H^{s,t}_{\mathrm{mot}} \cong H^{s,t}_{\acute{e}t}$ when $t \geq s$.

$E_2^{s,t} = 0$ when $s \geq 3$.

$E_2^{2,t} \cong \mathbb{Z}_p/\mathrm{num}(\zeta^*_{\mathbb{F}}(1-t))$.

$E_2^{1,t} \cong \mathbb{Z}_p^{\mathrm{ord}_{s=1-t}\zeta_{\mathbb{F}}(s)} \oplus \mathbb{Z}_p/\mathrm{denom}(\zeta^*_{\mathbb{F}}(1-t))$.

$E_2^{0,t} = 0$ when $t > 0$.

$2t - s$

# Dirichlet $L$-functions

### Definition

A **Dirichlet character** is a group homomorphism $\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$. The **Dirichlet $L$-function** attached to $\chi$ is:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}, \qquad \chi(n) = 0 \text{ if } (n,N) \neq 1.$$

- When $\chi$ is non-trivial, $L(s,\chi)$ converges when $\mathrm{Re}(s) > 0$ and admits analytic continuation to $\mathbb{C}$.
- Special values $L(1-n,\chi) \in \mathbb{Q}(\mathrm{Im}\,\chi)$. Simple zeros at $s = 1-n$ when $(-1)^n \neq \chi(-1)$.
- Functional equation: $\widehat{L}(s,\chi) = \widehat{L}(1-s,\chi^{-1})$.
- Notice $(\mathbb{Z}/N)^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Dirichlet $L$-functions are special cases of **Artin $L$-functions** $L(s,\rho)$ attached to Galois representations of number fields $\rho\colon \mathrm{Gal}(\mathbb{F}/\mathbb{K}) \to \mathrm{GL}_d(\mathbb{L})$.

# Goal of the project

## Goal

Formulate a Quillen-Borel Theorem and a Quillen-Lichtenbaum Conjecture for Dirichlet (or more generally Artin) $L$-functions.

## Slogan

For the Artin $L$-function $L(s,\rho)$ attached to a Galois representation $\rho\colon G = \mathrm{Gal}(\mathbb{F}/\mathbb{K}) \to \mathrm{GL}_d(\mathbb{L})$, its order of vanishing and special value at $s = 1-n$ are computed by $G$-equivariant algebraic $K$-groups of $\mathcal{O}_{\mathbb{F}}$ "<u>with coefficients in the representation $\rho$</u>".

Recall non-equivariantly, homotopy groups of a spectrum $X$ with coefficients in an abelian group $A$ is defined to be

$$\pi_n(X;A) := \pi_n(X \otimes M(A)),$$

where $M(A)$ is the Moore spectrum attached to $A$.

# Rational equivariant algebraic $K$-theory of number fields

# Quillen-Borel for Artin $L$-functions

**Theorem (Gross)**

*Consider the Artin $L$-function $L(s,\rho)$ attached to a Galois representation $\rho: G = \mathrm{Aut}(\mathbb{F}/\mathbb{K}) \to \mathrm{GL}_d(\mathbb{L})$. Denote the $\mathbb{L}$-vector space $\mathbb{L}^{\oplus d}$ with the associated $G$-action by $\underline{\rho}$. Then*

$$\mathrm{ord}_{s=1-n} L(s,\rho) = \dim_{\mathbb{L}} \left[ K_{2n-1}(\mathcal{O}_{\mathbb{F}}) \otimes_{\mathbb{Z}} \underline{\rho} \right]^G .$$

This result can be reformulated as:

**Corollary (Z.)**

*Notations as above. Then*

$$\mathrm{ord}_{s=1-n} L(s,\rho) = \dim_{\mathbb{L}} \pi_{2n-1} \left[ \left( K(\mathcal{O}_{\mathbb{F}}) \otimes M\left(\underline{\rho}\right)\right)^{hG} \right],$$

*where $M(\underline{\rho})$ is the $G$-**equivariant Moore spectrum** attached to $\rho$.*

# Moore spectra

## Definition

Let $A$ be an abelian group. The **Moore spectrum** attached to $A$ is the unique connective spectrum $M(A)$ such that

$$H_*(M(A);\mathbb{Z}) = \begin{cases} A, & * = 0; \\ 0, & \text{else.} \end{cases}$$

## Examples

- When $A = \mathbb{Z}$, $M(A)$ is the sphere spectrum $S^0$.
- When $A = C_n$, $M(A)$ is the cofiber of $S^0 \xrightarrow{n} S^0$.
- $M(A \oplus B) \simeq M(A) \vee M(B)$.

## Caution

The assignment $A \mapsto M(A)$ is NOT functorial in general.

# Rational equivariant Moore spectra

## Proposition

*Let $G$ be a finite group and $V$ be a $\mathbb{Q}$-vector space. Then any $G$-action on $V$ can be uniquely lifted to a $G$-action on the Moore spectrum $M(V)$, such that $H_0(M(V);\mathbb{Z})$ is $G$-equivariantly isomorphic to $V$.*

## Notation

Denote by $M(\underline{\rho})$ the $G$-equivariant Moore spectrum attached to the representation $\rho\colon G \to \operatorname{Aut}_{\mathbb{L}}(V)$.

## Proof of Quillen-Borel for Artin $L$-functions.

$$\begin{aligned}
\operatorname{ord}_{s=1-n} L(s,\rho) &= \dim_{\mathbb{L}} \left[K_{2n-1}(\mathcal{O}_{\mathbb{F}}) \otimes_{\mathbb{Z}} \underline{\rho}\right]^G \\
&= \dim_{\mathbb{L}} \left[\pi_{2n-1}\left(K(\mathcal{O}_{\mathbb{F}}) \otimes M(\underline{\rho})\right)\right]^G \\
&= \dim_{\mathbb{L}} \pi_{2n-1}\left[\left(K(\mathcal{O}_{\mathbb{F}}) \otimes M(\underline{\rho})\right)^{hG}\right].
\end{aligned}$$

$\square$

# Quillen-Lichtenbaum for Dirichlet $L$-functions

# Statement

## Conjecture

*Let* $\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ *be a Dirichlet character. Suppose* $\chi(-1) = (-1)^n$, *then the following holds up to powers of certain "bad" primes:*

$$\mathrm{Norm}(L(1-n,\chi)) = \frac{\#\pi_{2n-2}\left[\left(K(\mathbb{Z}[\zeta_N]) \otimes M(\underline{\mathcal{O}_\chi})\right)^{(\mathbb{Z}/N)^\times}\right]}{\#\pi_{2n-1}\left[\left(K(\mathbb{Z}[\zeta_N]) \otimes M(\underline{\mathcal{O}_\chi})\right)^{(\mathbb{Z}/N)^\times}\right]_{\mathrm{tors}}}.$$

Here $M(\underline{\mathcal{O}_\chi})$ is an integral $(\mathbb{Z}/N)^\times$-equivariant Moore spectrum attached to the character $\mathcal{O}_\chi$.

$$\begin{tikzcd} \chi\colon (\mathbb{Z}/N)^\times \arrow[r, two heads, "\phi_\chi"] \arrow[rr, bend right, "\mathcal{O}_\chi"'] & C_m \arrow[r, "\psi_m"] & (\mathbb{Z}[\zeta_m])^\times \arrow[r, hook] & \mathbb{C}^\times. \end{tikzcd}$$

First question: Does $M(\underline{\mathcal{O}_\chi})$ exist?

# Steenrod's question

**Question (Steenrod)**

*Let $A$ be an abelian group with a $G$-action. Is there a $G$-action on the Moore spectrum $M(A)$ such that the induced $G$-action on $H_0(M(A);\mathbb{Z})$ is isomorphic to the prescribed $G$-action on $A$?*

**Answer**

No in general to Steenrod's question. Carlsson has constructed a counter-example for every group of the form $C_p \times C_p$.

**Theorem (Z.)**

*For any abelian character $\chi: G \to \mathbb{C}^\times$ of a finite group $G$, an integral $G$-equivariant Moore spectrum $M(\underline{\mathcal{O}_\chi})$ exists as a finite $G$-CW spectrum. (Caution: no uniqueness in this case.)*

# Twisted Thomason spectral sequence

Let's now adapt the strategy for classical QLC to Dirichlet $L$-functions.

- {Galois representations} $\simeq$ {locally constant sheaves on étale site}
- Iwasawa theory: $L_p^*(1-n,\chi) \rightsquigarrow H^*_{ét}(\mathbb{Z}[1/(pN)], \underline{\chi} \otimes \mathbb{Z}_p(n))$ ,

  where $\underline{\chi}$ is the locally constant sheaf on $\mathbb{Z}[1/N]_{ét}$ corresponding to $\chi$.

**Question**

*Is there a $\chi$-twisted version of the Thomason spectral sequence?*

**Theorem (Elmanto-Z.)**

*Let $f\colon X \to Y$ be a finite $G$-étale cover of schemes, $\rho\colon G \to \mathrm{Aut}(A)$ be a Galois representation. Suppose $A$ is flat over $\mathbb{Z}_p$ and the $G$-equivariant Moore spectrum $M(\underline{\rho})$ exists. Then there is a spectral sequence:*

$$E_2^{s,2t} = H^s_{ét}(Y, \underline{\rho}(t)) \Longrightarrow \pi_{2t-s}\left[\left(L_{K(1)}K(X) \otimes M(\underline{\rho})\right)^{hG}\right].$$

# Another approach: Bootstrap from the classical QLC

## Proposition

*Let $\mathbb{F}/\mathbb{Q}$ be an abelian extension. Then $\mathbb{F} \subseteq \mathbb{Q}(\zeta_N)$ for some $N$ by the Kronecker-Weber Theorem. The Dedekind zeta function $\zeta_{\mathbb{F}}(s)$ factors as a product of Dirichlet/Artin $L$-functions:*

$$\zeta_{\mathbb{F}}(s) = \prod_{\substack{\chi:(\mathbb{Z}/N)^\times \to \mathbb{C}^\times \\ \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{F}) \subseteq \ker \chi}} L(s,\chi) = \prod_{\chi:\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \to \mathbb{C}^\times} L(s,\chi).$$

For example, let $\sigma: (\mathbb{Z}/4)^\times = C_2 \to \mathbb{C}^\times$ be the sign character. Then we have

$$\zeta_{\mathbb{Q}(i)}(s) = \zeta_{\mathbb{Q}}(s) L(s,\sigma).$$

Recall the classical QLC relates $\zeta_{\mathbb{Q}(i)}(s)$ and $\zeta_{\mathbb{Q}}(s)$ with $K(\mathbb{Z}[i])$ and $K(\mathbb{Z})$, respectively.

# QLC for $L(s, \sigma)$

## Proposition

*There is a cofiber sequence of algebraic $K$-theory spectra:*

$$K(\mathbb{Z}) \xrightarrow{f} K(\mathbb{Z}[i]) \longrightarrow \left(K(\mathbb{Z}[i]) \otimes S^{1-\sigma}\right)^{C_2},$$

*where $f$ is the induced by the extension of rings $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$.*

## Lemma

*The map $f$ induces injections in homotopy groups when $2$ is inverted.*

## Theorem (Elmanto-Z.)

*QLC holds for the Dirichlet $L$-function $L(s, \sigma)$ up to powers of $2$ if we choose the integral equivariant Moore spectrum $M(\underline{\mathcal{O}_\sigma})$ to be $S^{1-\sigma}$.*

### Proof of the Proposition.

The representation sphere $S^{\sigma-1}$ sits in a $C_2$-equivariant cofiber sequence:

$$S^{\sigma-1} \longrightarrow (C_2)_+ \longrightarrow (C_2/C_2)_+ \simeq S^0.$$

Mapping the sequence above into $K(\mathbb{Z}[i])$ and then taking $C_2$-fixed points yield the cofiber sequence. □

### Proof of the Lemma.

The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ is a syntomic extension. From this, we obtain a transfer map $\mathrm{tr}\colon K(\mathbb{Z}[i]) \to K(\mathbb{Z})$ such that $(\mathrm{tr} \circ f)_* = 2 \cdot -$ on $K_*(\mathbb{Z})$. □

### Two key ingredients

- $K(\mathbb{Z}[i])$ is a $C_2$-spectral Mackey functor.
- $C_2$-CW spectrum structure on the equivariant Moore spectrum $S^{1-\sigma}$.

# In progress

## Goal

Adapt this proof to general Dirichlet $L$-functions.

## Current status

- We have verified QLC for Dirichlet $L$-functions $L(s,\chi)$, where the image of $\chi$ is a cyclic group of prime power. In this case the equivariant Moore spectrum $M(\underline{\mathcal{O}_\chi})$ has two equivariant cells. So the same argument for $L(s,\sigma)$ applies.
- More generally, we need to compute the Bredon homology and AHSS:

$$H_*^{(\mathbb{Z}/N)^\times}\left(M(\underline{\mathcal{O}_\chi}); \underline{K}_*(\mathbb{Z}[\zeta_N])\right) \Longrightarrow \pi_*^{(\mathbb{Z}/N)^\times}\left(K(\mathbb{Z}[\zeta_N]) \bigwedge M(\underline{\mathcal{O}_\chi})\right).$$

- Also, we are currently computing $\mathrm{RO}(C_2)$-graded equivariant algebraic $K$-groups of $\mathbb{Z}[i]$ at prime 2.

# Thank you!