Twisted Real Quasi-elliptic cohomology

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December 16, 2022

Overview

Plan.

- Quasi-elliptic cohomology
  - Definition, Loop space construction
  - Power operation
  - Moduli problems
- Twisted Real Quasi-elliptic cohomology
  - Construction: Real loop space, twist, etc
  - Definition, examples, properties
- the Real Tate curve and related moduli problems
- Power operation
  - Power operation for Freed-Moore K-theory
  - Power operation for Real Quasi-elliptic cohomology

Quasi-elliptic cohomology

Explicit Definition

$$QEll^{\bullet}_{G}(X) := \prod_{g \in \pi_0(G^{tor} /\!/ G)} K^{\bullet}_{\Lambda_G(g)}(X^g)$$

- $\pi_0(G^{tor} /\!/ G)$: a set of representatives of $G-$conjugacy classes in G^{tor};
- $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$;
- $x \cdot [a, t] = x \cdot a$, for all $[a, t] \in \Lambda_G(g)$, $x \in X^g$.

$QEll^0_G(X)$ is an $\mathbb{Z}[q^{\pm}]-$algebra

$$1 \longrightarrow C_G(g) \longrightarrow \Lambda_G(g) \xrightarrow{\pi} \mathbb{T} \longrightarrow 0$$

$$\mathbb{Z}[q^{\pm}] = K^0_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\mathrm{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^{\bullet}_{G}(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^{\bullet}_{Tate}(X /\!/ G).$$

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Representation theory

Restriction map: $R\mathrm{G} \longrightarrow R\mathrm{H}$;

Equivariant K-theory

Restriction map: $K^{\bullet}_{\mathrm{G}}(X) \longrightarrow K^{\bullet}_{\mathrm{H}}(X)$;

Quasi-elliptic cohomology

Restriction map: $QEll^{\bullet}_{\mathrm{G}}(X) \longrightarrow QEll^{\bullet}_{\mathrm{H}}(X)$;

Basic Properties of Quasi-elliptic cohomology

Representation theory

Restriction map: $RG \longrightarrow RH$;

Induction map: $RH \longrightarrow RG$.

Equivariant K-theory

Restriction map: $K^{\bullet}_{G}(X) \longrightarrow K^{\bullet}_{H}(X)$;

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Restriction map: $RG \longrightarrow R\mathrm{H}$;

Induction map: $R\mathrm{H} \longrightarrow RG$.

$RG \otimes R\mathrm{H} \longrightarrow R(\mathrm{G} \times \mathrm{H})$.

Equivariant K-theory

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Induction map: $K^\bullet_{\mathrm{H}}(X) \longrightarrow K^\bullet_{\mathrm{G}}(X)$;

Künneth map: $K^\bullet_{\mathrm{G}}(X) \otimes K^\bullet_{\mathrm{H}}(Y) \longrightarrow K^\bullet_{\mathrm{G}\times\mathrm{H}}(X \times Y)$;

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Change-of-group isomorphism: $K^{\bullet}_{\mathrm{G}}(Y \times_{\mathrm{H}} \mathrm{G}) \xrightarrow{\cong} K^{\bullet}_{\mathrm{H}}(Y)$;

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Transfer ideal

Motivating Example: K-theory

$$I_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \text{Image}[I^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : K_{\Sigma_i\times\Sigma_j}(\text{pt}) \longrightarrow K_{\Sigma_N}(\text{pt})].$$

I_{tr} is the smallest ideal such that the quotient

$$P_N/I_{tr} : K(\text{pt}) \xrightarrow{P_N} K_{\Sigma_N}(\text{pt}) \to K_{\Sigma_N}(\text{pt})/I_{tr}$$

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Relation between elliptic cohomology and loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^{\infty}(\mathbb{T}, X),$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

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What is "Loop"?

Review: Free Loop Space

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$\mathbb{T}-$action: $\gamma \cdot t = (s \mapsto \gamma(s+t))$.

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$LG \rtimes \mathbb{T}-$action: $\gamma \cdot (\delta, t) = (s \mapsto \gamma(s+t) \cdot \delta(s+t))$.

$$(\delta_1, t_1) \cdot (\delta_2, t_2) = (s \mapsto \delta_1(s)\delta_2(s+t_1), t_1 + t_2).$$

Interpretation of the $LG \rtimes \mathbb{T}-$action

LG: the group of gauge transformations.

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$LG \rtimes \mathbb{T}$: the extended gauge group

$$\begin{array}{ccc} G \times \mathbb{T} & \xrightarrow{(g,s)\mapsto(\delta(s)g,s+t)} & G \times \mathbb{T} \\ \downarrow & & \downarrow \\ \mathbb{T} & \xrightarrow{s\mapsto s+t} & \mathbb{T} \end{array}$$

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$LG \rtimes \mathbb{T}$: act on loops

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The Answer: What is "Loop"?

New Definition of Equivariant loops $Loop(X/\!\!/G)$ [Rezk]

Objects:

$$\mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X$$

- π : principal G−bundle over $\mathbb{T}$
- f: G−equivariant;

Morphism (α, t) : $\{ \mathbb{T} \xleftarrow{\pi} P' \xrightarrow{f'} X \} \longrightarrow \{ \mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X \}$:

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Relation with Bibundles

$Bibun(\mathbb{T}/\!\!/*, X/\!\!/\mathrm{G})$

same objects;

morphisms: (α, Id). No rotations.

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$$\Lambda(X /\!\!/ G) \cong \coprod_{g \in \pi_0(G^{tor} /\!\!/ G)} X^g /\!\!/ \Lambda_G(g)$$

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Power operation of equivariant cohomology theories

Power operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.

Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

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$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau).$

Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)}).$

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Quasi-elliptic cohomology has power operations

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X/\!\!/G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

The Elliptic Power Operation [Huan]

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in\pi_0((G\wr\Sigma_n)^{tor}/\!\!/(G\wr\Sigma_n))} \mathbb{P}_{(\underline{g},\sigma)}:$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in\pi_0((G\wr\Sigma_n)^{tor}/\!\!/(G\wr\Sigma_n))} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\Lambda}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)})$$

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The Power Operation $\mathbb{P}_n$

Why is $\mathbb{P}_n$ good?

The construction can be generalized to other cohomology theories.

Uniquely extends to the **stringy power operation** of Tate K-theory.

Elliptic: reflect the geometric structure of the Tate curve.

Example ($G = e$)

$QEll^\bullet_G(X) = K^\bullet_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) =$
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Classification problems on the formal group

[Stricklands, Hopkins-Kuhn-Ravenel, 1990s]

	Complex K-theory	Morava E-theory E_n
Formal group	$\mathbb{G}_m$	$\mathbb{G}_u$
$\mathrm{Hom}(A^*, \mathbb{G})$	RA	$E_n^0(BA)$
Subgroups	$R\Sigma_{p^k}/I_{tr}$	$E_n^0(B\Sigma_{p^k})/I_{tr}$
A^*-Level structures	RA/I_A	$E_n^0(BA)/I_A$

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Classification problems on generalized $E-$theories

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	Quasi-elliptic cohomology $K^*_{orb}(\Lambda(-))$	$E^*_n(\mathcal{L}^h(-))$
Formal group	$\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}$	$\mathbb{G}_u \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h$
$\mathrm{Hom}(A^*, \mathbb{G})$	$K^*_{orb}(\Lambda(\mathrm{pt}/\!\!/A))$	$E^0_n(\mathcal{L}^h BA)$
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Twisted Real Quasi-elliptic cohomology

Expectation

(i) Defined to be a K-theory of a Real loop space

(ii) an equivariant elliptic cohomology associated to the Tate curve

(iii) Right relation to the moduli problems

$$\text{the cohomology theory} \xleftarrow{\text{a power operation}} \text{the Tate curve}$$

The right K-theory: Freed-Moore K-theory.

Combine the Realness, the twists, the equivariance geometrically.

Basic Set Up

(i) We start from a groupoid $\mathfrak{X}$.

(ii) An involution given by a double cover $\mathfrak{X} \to \hat{\mathfrak{X}}$.

(iii) A double cover $\Lambda(\mathfrak{X}) \to \hat{\Lambda}(\hat{\mathfrak{X}})$.

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(iv) $$\text{A twist on } \hat{\mathfrak{X}} \xrightarrow{\text{Real loop transgression}} \text{A twist on } \hat{\Lambda}(\hat{\mathfrak{X}})$$

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(iv) More generally, if $\hat{\mathrm{G}} = \mathrm{G} \rtimes \mathbb{Z}_2$ and X is a $\hat{\mathrm{G}}$-space,

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$$\mathbb{T} \xleftarrow{proj} P \xrightarrow{f} X$$

- $proj$: principal G−bundle over $\mathbb{T}$
- f: G−equivariant;

(ii) Morphism $(\alpha, (t, n))$:

$$\{ \mathbb{T} \xleftarrow{proj} P' \xrightarrow{f'} X \} \longrightarrow \{ \mathbb{T} \xleftarrow{proj} P \xrightarrow{f} X \}:$$

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Real Equivariant loops $Loop^R(X /\!\!/ G)$ [Huan, Young]

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The subgroupoid of constant Real Loops

Constant Real Loops

$$\{ \mathbb{T} \xleftarrow{proj} P_g \xrightarrow{f} X \mid f \text{ constant } \} = X^g$$

Question: What is the Real version of $\Lambda(X /\!\!/ G)$?

The Real centralizer of $g \in \hat{G}$

$$C^R_{\hat{G}}(g) = \{\omega \in \hat{G} \mid \omega g^{\pi(\omega)} \omega^{-1} = g\} \leqslant \hat{G}.$$

The enhanced Real stabilizer of $g \in G$

$$\Lambda^R_{\hat{G}}(g) := \left(\mathbb{R} \rtimes_\pi C^R_{\hat{G}}(g)\right) / \langle(-1, g)\rangle.$$

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$\Lambda^{\mathrm{ref}}_{\pi}(X/\!\!/\hat{\mathrm{G}})$ is the quotient groupoid $\Lambda(X/\!\!/\mathrm{G})/\!\!/(\iota_\omega, \Theta_\omega)$.

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$\Lambda\hat{\mathfrak{X}}$: quotient loop groupoid of $\mathfrak{X}$

Objects: $(x, \gamma) \in \hat{\mathfrak{X}}_0 \times Aut_{\hat{\mathfrak{X}}}(x)$;

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Involution on $\hat{\Lambda}\hat{\mathfrak{X}}$

objects: $(x, \gamma) \mapsto (x\omega, \omega^{-1}\gamma^{-1}\omega)$

morphisms: $(g, t) \mapsto (\omega^{-1}g^{-1}\omega, -t)$

G finite. Twist $QEll^{\bullet}_{G}(-)$ by $\alpha \in H^3(BG; U(1))$.

Transgression $\tau : H^3(BG; U(1)) \longrightarrow H^2(\mathrm{Map}(S^1, BG); U(1))$

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Real Transgression $\tilde{\tau}^{\mathrm{ref}}_{\pi} : H^3(B\hat{\mathrm{G}}; U(1)) \longrightarrow H^{2+\pi}(\mathrm{Map}(S^1, B\mathrm{G}); U(1))$

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Definition

$$QEllR^{\bullet+\hat{\alpha}}(X/\!\!/\mathrm{G}) = KR^{\bullet+\tilde{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}(\Lambda(X/\!\!/\mathrm{G})),$$

There is a $KR^{\bullet}_{\mathbb{T}}(\mathrm{pt})$-module isomorphism

$$QEllR^{\bullet+\hat{\alpha}}(X/\!\!/\mathrm{G}) \simeq \prod_{g\in\pi_0(\mathrm{G}/\!\!/_R\hat{\mathrm{G}})} {}^{\pi}K^{\bullet+\tilde{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}_{\Lambda^R_{\hat{\mathrm{G}}}(g)}(X^g).$$

More explicitly,

$$\begin{aligned} &QEllR^{\bullet+\hat{\alpha}}(X/\!\!/\mathrm{G}) \\ \simeq &\prod_{g\in\pi_0(\mathrm{G}/\!\!/\mathrm{G})_{-1}} KR^{\bullet+\tilde{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}_{\Lambda_{\mathrm{G}}(g)}(X^g) \times \prod_{g\in\pi_0(\mathrm{G}/\!\!/\mathrm{G})_{+1}/\mathbb{Z}_2} K^{\bullet+\tau(\alpha)}_{\Lambda_{\mathrm{G}}(g)}(X^g). \end{aligned}$$

Definition

$$QEllR^{\bullet+\hat{\alpha}}(X /\!\!/ G) = KR^{\bullet+\tilde{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}(\Lambda(X /\!\!/ G)),$$

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Examples

$\mathrm{G} = \{e\}$ and $\hat{\mathrm{G}} = \mathbb{Z}_2$

(i)
$$QEll^{\bullet}(X) \simeq K^{\bullet}_{\mathbb{T}}(X) \simeq K^{\bullet}(X)[q^{\pm 1}].$$

(ii) If X is a compact $\hat{\mathrm{G}}$-manifold,
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Properties of the theory

Recovering complex quasi-elliptic cohomology

- $\mathfrak{X}$: a groupoid;
- $\mathrm{G} = \{e\}$, $\hat{\mathrm{G}} = \mathbb{Z}_2$
- $\hat{\mathfrak{X}} = \mathfrak{X} \sqcup \mathfrak{X}$.

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Change-of-group isomorphism

$$\rho^{\hat{\mathrm{G}}}_{\hat{\mathrm{H}}} : QEllR^{\bullet}_{\mathrm{G}}(X \times_{\hat{\mathrm{H}}} \hat{\mathrm{G}}) \xrightarrow{\mathrm{Res}} QEllR^{\bullet}_{\mathrm{H}}(X \times_{\hat{\mathrm{H}}} \hat{\mathrm{G}}) \xrightarrow{j^*} QEllR^{\bullet}_{\mathrm{H}}(X)$$

is an isomorphism.

Induction map

$$\mathrm{RInd}^{\mathrm{G}}_{\mathrm{H}} : KR^{\bullet}_{\mathrm{H}}(X) \xrightarrow{\sim} KR^{\bullet}_{\mathrm{G}}(X \times_{\hat{\mathrm{H}}} \hat{\mathrm{G}}) \xrightarrow{f_!} KR^{\bullet}_{\mathrm{G}}(X).$$

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Künneth maps

$$C^R_{\hat{\mathrm{G}}\times_{\mathbb{Z}_2}\hat{\mathrm{H}}}(g,h) \simeq C^R_{\hat{\mathrm{G}}}(g) \times_{\mathbb{Z}_2} C^R_{\hat{\mathrm{H}}}(h),$$

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$$C^n(B\hat{\mathrm{G}}) \times C^n(B\hat{\mathrm{H}}) \to C^n(B(\hat{\mathrm{G}} \times_{\mathbb{Z}_2} \hat{\mathrm{H}})), \qquad (\hat{\alpha},\hat{\beta}) \mapsto \hat{\alpha}\hat{\beta}.$$

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$$QEllR^{\bullet+\hat{\alpha}}_{\mathrm{G}}(X) \hat{\otimes}_{KR^\bullet_{\mathbb{T}}(\mathrm{pt})} QEllR^{\bullet+\hat{\beta}}_{\mathrm{H}}(Y) := \prod_{\substack{g\in\pi_0(\mathrm{G}/\!\!/_R\hat{\mathrm{G}}) \\ h\in\pi_0(\mathrm{H}/\!\!/_R\hat{\mathrm{H}})}} {}^{\pi}K^{\bullet+\tilde{\tau}^{\mathrm{ref}}_{\pi}(\hat{\alpha})}_{\Lambda^R_{\hat{\mathrm{G}}}(g)}(X^g) \otimes_{KR^\bullet_{\mathbb{T}}(\mathrm{pt})} {}^{\pi}K^{\bullet+\tilde{\tau}^{\mathrm{ref}}_{\pi}(\hat{\beta})}_{\Lambda^R_{\hat{\mathrm{H}}}(h)}(Y^h).$$

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= &QEllR^{\bullet+\hat{\alpha}\hat{\beta}}_{\mathrm{G}\times\mathrm{H}}(X\times Y).
\end{aligned}$$

Relation with the Real Tate curve

$T[N]$ and $QEll^{\bullet}(-)$

$$T[N] \simeq \mathrm{Hom}(\mathbb{Z}_N^*, \mathrm{Tate}(q)) \simeq \mathrm{Spec}\,(QEll^0_{\mathbb{Z}_N}(\mathrm{pt}))$$

$\mathrm{G} = \mathbb{Z}_N$ and $\hat{\mathrm{G}} = D_{2N}$

The involution of $\pi_0(\mathrm{G}/\!\!/\mathrm{G})$ induced by $\hat{\mathrm{G}}$ is trivial.

The group inverse on $Tate(q) \quad \sim \quad (V \mapsto \overline{V})$

On $\mathrm{Hom}(\mathbb{Z}_N^*, \mathrm{Tate}(q))$

$$(\mathbb{Z}_N^* \xrightarrow{f} \mathrm{Tate}(q)) \quad \mapsto \quad (\mathbb{Z}_N^* \xrightarrow{\mathrm{Ad}^*_\omega} \mathbb{Z}_N^* \xrightarrow{f} \mathrm{Tate}(q) \xrightarrow{(-)^{-1}} \mathrm{Tate}(q)).$$

The homotopy fixed points of this $\mathbb{Z}_2$-action on $QEll^0_{\mathbb{Z}_N}(\mathrm{pt})$

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Set up for the Power operation

Real Structure on the Wreath product $G \wr \Sigma_N$

$\hat{G}$ is a Real structure on G.

$$\widehat{G \wr \Sigma_N} := \{(\underline{g}; \sigma) \in \hat{G} \wr \Sigma_N \mid \pi(g_i) = \pi(g_j) \text{ for all } i, j\}$$

$$\pi : \widehat{G \wr \Sigma_N} \to \mathbb{Z}_2, \quad (\underline{g}; \sigma) \mapsto \pi(g_i).$$

Twist

$$\wp_N : C^{n+\pi}(B\hat{G}) \to C^{n+\pi}(B\widehat{G \wr \Sigma_N})$$

$$\wp_N(\hat{\alpha})([a_1|\cdots|a_n]) = \prod_{j=1}^{N} \hat{\alpha}([g_{1_j}|g_{2_{\sigma_1^{-1}(j)}}|g_{3_{(\sigma_1\sigma_2)^{-1}(j)}}|\cdots|g_{n_{(\sigma_1\cdots\sigma_{n-1})^{-1}(j)}}])$$

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The twist is the right one

Let $\hat{\alpha}, \hat{\beta} \in C^{n+\pi}(B\hat{G})$. The cochain operations $\{\wp_N\}_{N\geq 0}$ have the following properties:

(i) $\wp_0(\hat{\alpha}) = 1$ and $\wp_1(\hat{\alpha}) = \hat{\alpha}$.

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X: a $\hat{\mathrm{G}}$-space.

$V \to X$: a $\hat{\theta}$-twisted Real G-equivariant vector bundle,
$V^{\boxtimes N} \to X^{\times N}$: a $\wp_N(\hat{\theta})$-twisted Real $\mathrm{G} \wr \Sigma_N$-equivariant vector bundle.

$$P_N^{R,\hat{\theta}} : {}^{\pi}K^{\bullet+\hat{\theta}}_{\hat{\mathrm{G}}}(X) \to {}^{\pi}K^{\bullet+\wp_N(\hat{\theta})}_{\widehat{\mathrm{G}\wr\Sigma_N}}(X^{\times N}), \qquad V \mapsto V^{\boxtimes N}.$$

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Power operations for Freed-Moore K-theory [Huan, Young]

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(ii) The (external) product of two operations is

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Power operation of twisted Real quasi-elliptic cohomology?

$$\mathbb{P}_N^{R,\hat{\alpha}} = \prod_{(\underline{g},\sigma)\in\pi_0((G\wr\Sigma_N)^{tor}/\!\!/_R\widehat{G\wr\Sigma_N})} \mathbb{P}^{R,\hat{\alpha}}_{(\underline{g};\sigma)}$$

$$\mathbb{P}^{R,\hat{\alpha}}_{(\underline{g};\sigma)} : QEllR_G^{\bullet+\hat{\alpha}}(X) \xrightarrow{U_R^*} KR^{\bullet+U_R^*(\tilde{\tau}_\pi^{ref}(\hat{\alpha}))}(\Lambda^1_{(\underline{g};\sigma)}(X)) \xrightarrow{(\ )_k^\Lambda}$$

$$KR^{\bullet+(\ )_k^\Lambda\circ U_R^*(\tilde{\tau}_\pi^{ref}(\hat{\alpha}))}(\Lambda^{var}_{(\underline{g};\sigma)}(X)) \xrightarrow{\boxtimes} KR^{\bullet+d}(d_{(\underline{g};\sigma)}(X)) \xrightarrow{f^*_{(\underline{g};\sigma)}}$$

$$KR^{\bullet+f^*_{(\underline{g};\sigma)}(d)}_{\Lambda_{G\wr\Sigma_N}(\underline{g};\sigma)}((X^{\times N})^{(\underline{g};\sigma)}),$$

where the twist is

$$d := \prod_k \prod_{(i_1,\dots,i_k)} (\ )_k^\Lambda \circ U_R^*(\tilde{\tau}_\pi^{ref}(\hat{\alpha}))_{g_{i_k}\cdots g_{i_1}},$$

If $\hat{\alpha}$ is trivial

$\{\mathbb{P}_N^{R,\hat{\alpha}}\}_{N\geq 0}$ gives a power operation for Real quasi-elliptic cohomology.

Power operation of twisted Real quasi-elliptic cohomology?

$$\mathbb{P}^{R,\hat{\alpha}}_N = \prod_{(\underline{g},\sigma)\in\pi_0((\mathrm{G}\wr\Sigma_N)^{\mathrm{tor}}/\!\!/_R\widehat{\mathrm{G}\wr\Sigma_N})} \mathbb{P}^{R,\hat{\alpha}}_{(\underline{g};\sigma)}$$

$$\mathbb{P}^{R,\hat{\alpha}}_{(\underline{g};\sigma)}: QEllR^{\bullet+\hat{\alpha}}_{\mathrm{G}}(X) \xrightarrow{U^*_R} KR^{\bullet+U^*_R(\tilde{\tau}^{\mathrm{ref}}_{\pi}(\hat{\alpha}))}(\Lambda^1_{(\underline{g};\sigma)}(X)) \xrightarrow{(\ )^{\Lambda}_k}$$

$$KR^{\bullet+(\ )^{\Lambda}_k\circ U^*_R(\tilde{\tau}^{\mathrm{ref}}_{\pi}(\hat{\alpha}))}(\Lambda^{\mathrm{var}}_{(\underline{g};\sigma)}(X)) \xrightarrow{\boxtimes} KR^{\bullet+d}(d_{(\underline{g};\sigma)}(X)) \xrightarrow{f^*_{(\underline{g};\sigma)}}$$

$$KR^{\bullet+f^*_{(\underline{g};\sigma)}(d)}_{\Lambda_{\mathrm{G}\wr\Sigma_N}(\underline{g};\sigma)}((X^{\times N})^{(\underline{g};\sigma)}),$$

where the twist is

$$d := \prod_k \prod_{(i_1,\ldots,i_k)} (\ )^{\Lambda}_k \circ U^*_R(\tilde{\tau}^{\mathrm{ref}}_{\pi}(\hat{\alpha}))_{g_{i_k}\cdots g_{i_1}},$$

If $\hat{\alpha}$ is trivial

$\{\mathbb{P}^{R,\hat{\alpha}}_N\}_{N\geq 0}$ gives a power operation for Real quasi-elliptic cohomology.

Power operation of twisted Real quasi-elliptic cohomology?

$$\mathbb{P}_N^{R,\hat{\alpha}} = \prod_{(\underline{g},\sigma)\in\pi_0((\mathrm{G}\wr\Sigma_N)^{\mathrm{tor}}/\!\!/_R\widehat{\mathrm{G}\wr\Sigma_N})} \mathbb{P}^{R,\hat{\alpha}}_{(\underline{g};\sigma)}$$

$$\mathbb{P}^{R,\hat{\alpha}}_{(\underline{g};\sigma)} : QEllR_{\mathrm{G}}^{\bullet+\hat{\alpha}}(X) \xrightarrow{U_R^*} KR^{\bullet+U_R^*(\tilde{\tau}_\pi^{\mathrm{ref}}(\hat{\alpha}))}(\Lambda^1_{(\underline{g};\sigma)}(X)) \xrightarrow{(\ )_k^\Lambda}$$

$$KR^{\bullet+(\ )_k^\Lambda\circ U_R^*(\tilde{\tau}_\pi^{\mathrm{ref}}(\hat{\alpha}))}(\Lambda^{\mathrm{var}}_{(\underline{g};\sigma)}(X)) \xrightarrow{\boxtimes} KR^{\bullet+d}(d_{(\underline{g};\sigma)}(X)) \xrightarrow{f^*_{(\underline{g};\sigma)}}$$

$$KR^{\bullet+f^*_{(\underline{g};\sigma)}(d)}_{\Lambda_{\mathrm{G}\wr\Sigma_N}(\underline{g};\sigma)}((X^{\times N})^{(\underline{g};\sigma)}),$$

where the twist is

$$d := \prod_k \prod_{(i_1,\ldots,i_k)} (\ )_k^\Lambda \circ U_R^*(\tilde{\tau}_\pi^{\mathrm{ref}}(\hat{\alpha}))_{g_{i_k}\cdots g_{i_1}},$$

If $\hat{\alpha}$ is trivial

$\{\mathbb{P}_N^{R,\hat{\alpha}}\}_{N\geq 0}$ gives a power operation for Real quasi-elliptic cohomology.

Power operation of twisted Real quasi-elliptic cohomology?

The left problem

$$\begin{array}{ccc} C^{\bullet+\pi}(B\hat{\mathrm{G}}) & \xrightarrow{\tilde{\tau}_{\pi}^{\mathrm{ref}}} & C^{\bullet-1}(\mathcal{L}_{\pi}^{\mathrm{ref}} B\hat{\mathrm{G}}) \\ \downarrow \wp_N & & \downarrow P_N \\ C^{\bullet+\pi}(B\widehat{\mathrm{G}\wr\Sigma_N}) & \xrightarrow{\tilde{\tau}_{\pi}^{\mathrm{ref}}} & C^{\bullet-1}(\mathcal{L}_{\pi}^{\mathrm{ref}} B\widehat{\mathrm{G}\wr\Sigma_N}) \end{array}$$

commutes?

Does P_N commute with the loop transgression?

Power operation of twisted Real quasi-elliptic cohomology?

The left problem

$$\begin{array}{ccc} C^{\bullet+\pi}(B\hat{\mathrm{G}}) & \xrightarrow{\tilde{\tau}^{\mathrm{ref}}_{\pi}} & C^{\bullet-1}(\mathcal{L}^{\mathrm{ref}}_{\pi} B\hat{\mathrm{G}}) \\ \wp_N \downarrow & & \downarrow P_N \\ C^{\bullet+\pi}(B\widehat{\mathrm{G}\wr\Sigma_N}) & \xrightarrow{\tilde{\tau}^{\mathrm{ref}}_{\pi}} & C^{\bullet-1}(\mathcal{L}^{\mathrm{ref}}_{\pi} B\widehat{\mathrm{G}\wr\Sigma_N}) \end{array}$$

commutes?

Does P_N commute with the loop transgression?

Thank you.

Some references

https://huanzhen84.github.io/zhenhuan/Huan-2022-SUSTech.pdf

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