The ordinary and motivic cohomology of $BPGL_n$

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Arithmetic and Topology, SUSTech

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Classifying spaces and characteristic classes

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- $G \mapsto BG$ is a functor from the category of topological groups to the homotopy category of topological spaces, i.e., for a homomorphism $G \to H$ of topological groups, there is a uniquely determined homotopy class of maps $BG \to BH$ satisfying certain compatibility conditions.

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Remark

In the above construction, one can let G be a simplicial group and replace geometric realization with the diagonal (of a bisimplicial set), the resulting simplicial set, also denoted by BG, is the classifying space of the simplicial group G.

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Example

$$H^*(BGL_n(\mathbb{C}); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_n].$$

The class c_i is the ith *Chern class*.

Motivic cohomology

Motivic cohomology

Fix a base field k and abelian group A. Motivic cohomology is a collection of functors, indexed by pairs (s, t) of non-negative integers, from $\mathbf{Sm}^k$, the category of smooth schemes over k, to the category $\mathscr{A}b$ of abelian groups

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$$\cdots \to H_M^{s,t}(-;A_0) \to H_M^{s,t}(-;A_1) \to H_M^{s,t}(-;A_2) \xrightarrow{\delta} H_M^{s+1,t}(-;A_0) \cdots .$$

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- Given a commutative unital ring R, the functors $H_M^{s,t}(-;R)$ collectively form a functor from $\mathbf{Sm}^k$ to $R\mathscr{A}lg^{*,*}$, the category of bi-graded, bi-commutative R-algebras:

$$H_M^{*,*}(-;R) : \mathbf{Sm}^k \to R\mathscr{A}lg^{*,*}.$$

Motivic cohomology

(last slide continued)

...

- (homotopy invariance, or $\mathbf{A}^1$-invariance) For the affine line $\mathbf{A}^1$, the canonical projection $X \times \mathbf{A}^1 \to X$ induces an isomorphism

$$H_M^{*,*}(X; A) \xrightarrow{\cong} H_M^{*,*}(X \times \mathbf{A}^1; A).$$

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$$H_M^{s,1}(X;\mathbb{Z}) = \begin{cases} \mathscr{O}^\times(X),\ s=1 \text{ (the invertible elements in } \mathscr{O}(X)), \\ \mathrm{Pic}(X),\ s=2 \text{ (the Picard group of } X), \\ 0,\ s\neq 1,2. \end{cases}$$

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Example (Milnor K-theory)

$H_M^{s,s}(\mathrm{spec}(k), A) = K_s^M(k) \otimes A$, where $K_s^M(k)$ are the Milnor K-groups of k.

Motivic cohomology

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For a strictly Hensel local scheme S over k, and an integer n prime to the characteristic of k, we have

$$H^{s,t}(\mathrm{spec}(S); \mathbb{Z}/n) = \begin{cases} \mu_n^{\otimes t}(S), & s = 0, \\ 0, & s \neq 0, \end{cases}$$

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In general, étale cohomology cannot be recovered from motivic cohomology. In particular, it is generally not $\mathbf{A}^1$-invariant.

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In general, étale cohomology cannot be recovered from motivic cohomology. In particular, it is generally not $\mathbf{A}^1$-invariant.

We would like to understand motivic cohomology in a homotopy-theoretical setting, like we did in the case of singular cohomology.

Motivic homotopy theory

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For instance, algebraic stacks are simplicial presheaves over schemes.

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Different Grothendieck topologies can be defined for the same category. Sometimes, one is "finer" than another.

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- **étale**: Strict Henselian rings, i.e., Henselian rings with separably closed residue fields.

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Let $\mathbf{Sm}^k_{Nis}$ be the site of smooth schemes over a field k with the Nisnevich topology. Define

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Therefore, the category of "motivic spaces" $\mathbf{Mot}^k_\bullet$ over k canonically has a structure of model category, i.e., notaions of weak equivalences, fibrations and cofibrations.
We denote the corrsponding homotopy category by $\mathbf{HMot}^k_\bullet$.

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$$B_{Nis}G = \mathrm{diag}(EG/G) \in Mot_\bullet^k$$

The simplicial Nisnevich presheaf $B_{Nis}G$ "classifies" Nisnevich G-torsors:

$$Torsor_{Nis}(-) \xrightarrow{\simeq} \mathbf{HMot}_\bullet^k(-, B_{Nis}G)$$

where $Torsor_{Nis}(X)$ denotes the isomorphism classes of Nisnevich G-torsors over X.

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which induces a pair of adjoint functors

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It turns out that the homotopy category $\mathbf{H}_{\mathbf{A}^1}\mathbf{Mot}_{\bullet}^{k}$ is much easier to work with than $\mathbf{HMot}_{\bullet}^{k}$.

However, the advantages come with costs. For instance, the natural transformation

$$Torsor_{\acute{e}t}(-) \to \mathbf{H}_{\mathbf{A}^1}\mathbf{Mot}_{\bullet}^{k}(-, BG)$$

is, in general, neither surjective nor injective.

Let $H_M^{*,*}(-;R)$ denote motivic cohomology with coefficients in a commutative unital ring R.

Motivic Eilenberg-Mac Lane spaces

Let $H_M^{*,*}(-;R)$ denote motivic cohomology with coefficients in a commutative unital ring R. For $0 \leq t \leq s$, we have the motivic Eilenberg-Mac Lane space $K(R(t),s)$ representing motivic cohomology, i.e.,

$$\mathbf{H}_{\mathbf{A}^1}\mathbf{Mot}_{\bullet}^{k}(-,K(R(t),s)) \cong H_M^{s,t}(-;R)$$

where $H_M^{*,*}(-;R)$ denotes the motivic cohomology with coefficients in R.

The motivic Steenrod reduced power operations

For p an odd prime, we have operations

$$\mathrm{P}^i : H_M^{s,t}(-;\mathbb{F}_p) \to H_M^{s+2i(p-1),t+i(p-1)}(-;\mathbb{F}_p)$$

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For $p = 2$, we have the motivic counterpart of the Steenrod squares as well.

The (complex) topological realization

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Passing to homotopy categories, we have

$$t^{\mathbb{C}} : \mathbf{H}_{\mathbf{A}^1}\mathbf{Mot}_\bullet^{\mathbb{C}} \to \mathbf{H}\mathcal{T}_\bullet.$$

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which we called *the cycle class map*. It generalizes the ordinary cycle class map

$$\mathrm{cl} : \mathrm{CH}^t(X) \otimes R = H_M^{2t,t}(X; R) \to H^{2t}(X(\mathbb{C}); R)$$

where X is a smooth complex algebraic variety.

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The cycle class map for BG

For a complex algebraic group G, we have $t^{\mathbb{C}}(BG) \cong BG(\mathbb{C})$, and therefore

$$\mathrm{cl}: H_M^{s,t}(BG; R) \to H^s(BG(\mathbb{C}); R)$$

and in particular,

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Example

Let $G = GL_n$ or SL_n. Then

$$\mathrm{cl}: CH^t(BGL_n) = H_M^{2t,t}(BGL_n; \mathbb{Z}) \to H^{2t}(BU_n; \mathbb{Z}),$$
$$\mathrm{cl}: CH^t(BSL_n) = H_M^{2t,t}(BSL_n; \mathbb{Z}) \to H^{2t}(BSU_n; \mathbb{Z}).$$

are isomorphisms.

The main theorems

On the Chow ring of $BPGL_n$

Let $PGL_n = GL_n/\mathbb{C}^\times$. The Chow ring and cohomology of $BPGL_n$ are difficult.

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- Rationally, we have isomorphisms

$$\mathrm{CH}^*(BPGL_n) \otimes \mathbb{Q} \cong \mathrm{CH}^*(BSL_n) \otimes \mathbb{Q},$$
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$$\begin{cases} H^1(BPGL_n;\mathbb{Z}) = H^2(BPGL_n;\mathbb{Z}) = 0, \\ H^3(BPGL_n;\mathbb{Z}) \cong \mathbb{Z}/n. \end{cases}$$

Review of the literature

Currently, the study of the cohomology and Chow ring of $BPGL_n$ is rather incomplete:

- (Kono and Mimura, 1971) The $\mathbb{F}_2$-module structure of $H^*(BPGL_{4k+2}; \mathbb{F}_2)$.
- (Toda, 1986), The $\mathbb{F}_2$-algebra structure of $H^*(BPGL_{4k+2}; \mathbb{F}_2)$ and $H^*(BPGL_4; \mathbb{F}_2)$.
- (Vavpetič and Viruel, 2005) Some results on $H^*(BPGL_p; \mathbb{F}_p)$ for a prime number p.
- (Vezzosi, 2000) An almost complete computation of $H^*(BPGL_3)$ and $\mathrm{CH}^*(BPGL_3)$.
- (Vistoli, 2007) A complete computation of $H^*(BPGL_3)$ and $\mathrm{CH}^*(BPGL_3)$, and an almost complete computation of $H^*(BPGL_p)$ and $\mathrm{CH}^*(BPGL_p)$ for $p > 3$ a prime number.

Review of the literature

However, none of the works listed above concerns the integral cohomology $H^*(BPU_n)$ for arbitrary n. Regarding this, we have

- (G, 2016) The ring structure of $H^*(BPGL_n)$ in dimensions less than 11.
- (G, 2019, 2020) A distinguished subring generated by p-torsion classes of $\mathrm{CH}^*(BPGL_n)$ and $H^*(BPGL_n)$.
- (G-Zhang-Zhang-Zhong, 2021) The p-local cohomology $H^k(BPGL_n)_{(p)}$ for $k < 2p+5$ and a prime number p.

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The induced map $B\Delta : BPGL_m \to BPGL_n$ plays a key role in what follows.

On the Chow ring of $BPGL_n$

Theorem (G, 2020)

Let p be an odd prime, and n a positive integer divisible by p. Then there are nontrivial p-torsion classes

$$\rho_{p,k}(n) \in \mathrm{CH}^{p^{k+1}+1}(BPGL_n), \quad y_{p,k}(n) = \mathrm{cl}(\rho_{p,k}(n)) \in H^{2p^{k+1}+2}(BPGL_n)$$

for $k \geq 0$,

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for $k \geq 0$, such that for $p \mid m \mid n$ and $\Delta : PGL_m \to PGL_n$, we have

$$B\Delta^*(\rho_{p,k}(n)) = \rho_{p,k}(m), \; B\Delta^*(y_{p,k}(n)) = y_{p,k}(m). \tag{1}$$

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Furthermore, suppose $r \geq 1$ satisfies $p^r \mid n$ and $p^{r+1} \nmid n$. Then there are injective ring homomorphisms

$$\mathbb{Z}[Y_k \mid 0 \leq k \leq 2r-1]/(pY_k) \hookrightarrow \mathrm{CH}^*(BPGL_n), \ Y_k \mapsto \rho_{p,k}(n), \tag{2}$$

$$\mathbb{Z}[Y_k \mid 0 \leq k \leq 2r-1]/(pY_k) \hookrightarrow H^*(BPGL_n), \ Y_k \mapsto y_{p,k}(n). \tag{3}$$

When there is no risk of ambiguity, we write $\rho_{p,k}$ and $y_{p,k}$ for $\rho_{p,k}(n)$ and $y_{p,k}(n)$, respectively.

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$$\begin{cases} \mathfrak{R}_M(n) = \mathbb{Z}[\rho_{p,k} \mid k \geq 0]/(p\rho_{p,k}) \subset \mathrm{CH}^*(BPGL_n), \\ \mathfrak{R}(n) = \mathbb{Z}[y_{p,k} \mid k \geq 0]/(py_{p,k}) \subset H^*(BPGL_n). \end{cases}$$

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Theorem (G, 2021)

Let p be an odd prime and $n > 1$ an integer with p-adic valuation $r > 0$. Then the homomorphisms $B\Delta^*$ restrict to isomorphisms

$$\begin{cases} B\Delta^* : \mathfrak{R}_M(n) \xrightarrow{\cong} \mathfrak{R}_M(p^r),\ \rho_{p,k}(n) \mapsto \rho_{p,k}(p^r), \\ B\Delta^* : \mathfrak{R}(n) \xrightarrow{\cong} \mathfrak{R}(p^r),\ y_{p,k}(n) \mapsto y_{p,k}(p^r). \end{cases}$$

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We are not yet able to determine the polynomial relations in $\{\rho_{p,k}\}_{k\leq i}$ or $\{y_{p,k}\}_{k\leq i}$ for $i \geq 2r$.

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However, for $r = 1$, we have the following

Theorem (G, 2020)

For p and odd prime, and $n > 0$ an integer satisfying $p \mid n$ and $p^2 \nmid n$, the classes $\rho_{p,k} \in \mathrm{CH}^*(BPGL_n)$ for $k = 0, 1, 2$, satisfy a nontrivial polynomial relation

$$\rho_{p,0}^{p^2+1} + \rho_{p,1}^{p+1} + \rho_{p,0}^{p}\rho_{p,2} = 0, \tag{4}$$

and similarly for $y_{p,k} \in H^*(BPGL_n)$, $k = 0, 1, 2$, we have

$$y_{p,0}^{p^2+1} + y_{p,1}^{p+1} + y_{p,0}^{p}y_{p,2} = 0. \tag{5}$$

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- the Beilinson-Lichtenbaum Conjecture/Theorem (Voevodsky, 2008):

$$H_M^{s,t}(X;\mathbb{Z}/m) \cong H_{\acute{e}t}^{s}(X;\mu_m^{\otimes t})$$

for $s \leq t$, where μ_m is the étale sheaf represented by

$$\mathrm{spec}(k[x]/(x^m - 1)).$$

The Beilinson-Lichtenbaum Conjecture/Theorem

The Beilinson-Lichtenbaum Conjecture/Theorem is a generalization of a fundamental result, the norm residue isomorphism theorem (or Bloch–Kato conjecture), which concerns an isomorphism between Milnor K-theory and étale cohomology of a field.

The "fundamental" classes

We construct "fundamental" classes

$$x_1 \in H^3(BPGL_n),\ \zeta_1 \in H_M^{3,2}(BPGL_n)$$

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Composing it with the connecting homomomorphim $H^2(-; \mathbb{Z}/n) \to H^3(-; \mathbb{Z})$, we obtain

$$[-, BPGL_n] \cong H^1(-; PGL_n) \to H^3(-; \mathbb{Z}) \cong [-, K(\mathbb{Z}, 3)],$$

and by Yoneda lemma we obtain a homotopy class of maps $BPGL_n \to K(\mathbb{Z}, 3)$, which is the "fundamental" class $x_1 \in H^3(BPGL_n)$.

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Recall the isomorphism

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Therefore, the connecting homomorphism $H^1_{ét}(-; PGL_n) \to H^2_{ét}(-; \mu_n)$ induces a morphism in $\mathbf{HMot}^{\mathbb{C}}_{Nis}$:

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which passes to a morphism in $\mathbf{HMot}^{\mathbb{C}}_{\bullet}$.

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Recall the Beilinson-Lichtenbaum Conjecture/Theorem:

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&H^2_{ét}(-;\mu_n)\\
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\cong &H_M^{2,2}(-;\mathbb{Z}/n) \ \ (\text{Beilinson-Lichtembaum})\\
\cong &\mathbf{H}_{\mathbf{A}^1}\mathbf{Mot}_\bullet^{\mathbb{C}}(-, K(\mathbb{Z}/n(2),2)).
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and an isomorphism in $\mathbf{HMot}_\bullet^{\mathbb{C}}$:

$$B^2\mu_n\to K(\mathbb{Z}/n(2),2).$$

The “fundamental” classes

Finally we obtain a morphism in $\mathbf{HMot}_{\bullet}^{\mathbb{C}}$:

$$BPGL_n \to K(\mathbb{Z}/n(2), 2) \to K(\mathbb{Z}(2), 3)$$

which represents the “fundamental” class $\zeta_1 \in H_M^{3,2}(BPGL_n)$.

The rest is topology

Recall that the cycle class map

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is a ring homomorphism that commutes with the Steenrod operations. In particular, we have the "topological fundamental class" $x_1 = \mathrm{cl}(\zeta_1)$. The rest of the proofs therefore involve mostly singular cohomology.

Thank You!